Understanding Jordan Normal Form

Brendan Cordy

In this article, we'll try to understand a linear operator on a complex vector space by breaking it up into smaller, simpler pieces, and discover the Jordan Normal Form.

Decomposing Linear Operators

Consider a finite dimensional vector space U over $\mathbb{C}$ with $dim(U) > 1$. It's always possible to write U as a direct sum of two spaces of smaller dimension. Take a basis for U, divide it into two nonempty subsets, and call the spaces generated by the subsets V and W. Then $U = V \oplus W$.

If $T : U \to U$ is a linear operator, we would like to cleverly choose a basis and divide it so that T sends every vector in V to V and every vector in W to W, making $T|_V$ and $T|_W$ well-defined linear operators on the subspaces V and W. If this is possible, our decomposition of U has broken the linear operator T into two simpler *non-interacting* pieces.

Definition: A linear operator $T : U \to U$ is called *decomposable* if $T = T|_V \oplus T|_W$ for nontrivial subspaces $V \subset U$ and $W \subset U$ with $U = V \oplus W$. If this is not possible, T is *indecomposable.*

Subspaces like V and W, which are closed under the action of T, are called *T-invariant subspaces.* If you've studied eigenvectors, then you already know a bit about invariant subspaces. Eigenspaces of T are always T-invariant, though not every invariant subspace is an eigenspace.

Example: Let $T : \mathbb{C}^3 \to \mathbb{C}^3$ be given by $T\langle z_1, z_2, z_3\rangle = \langle 3z_1 + z_3,\ 2z_2,\ z_1 - z_3\rangle$.

Observe that $V = span(\langle 0, 1, 0\rangle)$ is an T-invariant subspace. If we apply T to an arbitrary vector in V, we have $T(c\,\langle 0, 1, 0\rangle) = c\,T(\langle 0, 1, 0\rangle) = \langle 0, 2c, 0\rangle$, which is again in V. Thus, V is T-invariant. Note that the vector $\langle 0, 1, 0\rangle$ is an eigenvector of T for the eigenvalue $\lambda = 2$. In fact, any one dimensional T-invariant subspace is the span of an eigenvector.

If we take $W = span(\langle 1, 0, 0\rangle, \langle 0, 0, 1\rangle)$, we obtain another T-invariant subspace, but this time W is not an eigenspace. Let's check it.

$$T(c\,\langle 1, 0, 0\rangle + d\langle 0, 0, 1\rangle) = c\,T(\langle 1, 0, 0\rangle) + d\,T(\langle 0, 0, 1\rangle) = \langle 3c + d,\ 0,\ c - d\rangle \in W$$

Note that W is not an eigenspace of T since the vector $\langle 3c + d,\ 0,\ c - d\rangle$ is, in general, not a scalar multiple of $\langle c, 0, d\rangle$. Take $c = 1$ and $d = 1$ for example, then $T(\langle 1, 0, 1\rangle) = \langle 2, 0, 0\rangle$.

Since $V \oplus W = \mathbb{C}^3$, T is decomposable. We have $T = T|_V \oplus T|_W$. Combining a basis of V with a basis of W, we obtain a basis for $\mathbb{C}^3$ in which T can be written as a block diagonal matrix. In this case, consider the basis $\beta = \{\langle 0, 1, 0\rangle, \langle 1, 0, 0\rangle, \langle 0, 0, 1\rangle\}$.

$$[T]_\beta = \left[\begin{array}{c|cc} 2 & 0 & 0 \\ \hline 0 & 3 & 1 \\ 0 & 1 & -1 \end{array}\right]$$

The upper block is the matrix of $T|_V$ and the lower block is the matrix of $T|_W$, written in the bases that we amalgamated to form β. Note the zero entries in the upper-right and lower-left. Nonzero entries in the lower left would indicate T is sending vectors from V into W, while nonzero entries in the upper right would indicate T is sending vectors from W into V.

Exercise: Continue decomposing T by finding a T-invariant subspace of $T|_W$.

Example: Let $T : \mathbb{C}^3 \to \mathbb{C}^3$ be given by $T\langle z_1, z_2, z_3\rangle = \langle z_1 + z_2 + z_3,\ z_2 + z_3,\ z_3\rangle$.

Again, it's easy to find a one dimensional T-invariant subspace by inspection, $V = span(\langle 1, 0, 0\rangle)$.

This time, if we apply T to a vector in $W = span(\langle 0, 1, 0\rangle, \langle 0, 0, 1\rangle)$ we can see that W is not an invariant subspace.

$$T(c\,\langle 0, 1, 0\rangle + d\langle 0, 0, 1\rangle) = c\,T(\langle 0, 1, 0\rangle) + d\,T(\langle 0, 0, 1\rangle) = \langle c + d,\ c + d,\ d\rangle \notin W$$

Given a T-invariant subspace V, that subspace may or may not have a complementary subspace W which is also T-invariant. In this example, there is no such complementary invariant subspace (we'll show it later). You can try to find a T-invariant subspace W with $V \oplus W = \mathbb{C}^3$, but you will not succeed; the linear operator T is indecomposable.

What does the indecomposability of T tell us about how T can be written as a matrix? Let's consider a basis $\beta = \{\langle 1, 0, 0\rangle, \vec{w_1}, \vec{w_2}\}$ of $\mathbb{C}^3$. Since $T(\langle 1, 0, 0\rangle) = \langle 1, 0, 0\rangle$,

$$[T]_\beta = \begin{bmatrix} 1 & \# & \# \\ 0 & * & * \\ 0 & * & * \end{bmatrix}$$

If T is indecomposable, then for any vectors $\vec{w_1}$ and $\vec{w_2}$, the entires marked $\#$ in the matrix of T above cannot both be zero. If they were, then $span(\vec{w_1}, \vec{w_2})$ would be a complementary T-invariant subspace for V, and T could be decomposed.

Exercise: Recall that every linear operator T on a finite dimensional complex vector space has an eigenvector. Use this to show that if every T-invariant subspace had a complementary invariant subspace, then every linear operator would be diagonalizable.

If a pair of complementary T-invariant subspaces does exist, can we find them in some systematic way? The most natural subspaces that come with a linear operator are its kernel and image, and conveniently enough, these are both T-invariant. So that seems like a good place to start.

To show $ker(T)$ is a T-invariant subspace, take $\vec{v} \in ker(T)$ and apply T, to obtain $T(\vec{v})$. Is this vector still in $ker(T)$? Yes, since $T(T(\vec{v})) = T(\vec{0}) = \vec{0}$. Hence, $ker(T)$ is invariant under T.

Exercise: Show that $im(T)$ is also a T-invariant subspace.

Could it be that $U = ker(T) \oplus im(T)$, so that any T could be decomposed as $T|_{ker(T)} \oplus T|_{im(T)}$?

Example: Consider the linear operator $T : \mathbb{C}^3 \to \mathbb{C}^3$ given by $T\langle z_1, z_2, z_3\rangle = \langle z_1, 2z_1, z_1 + z_2\rangle$.

Observe that $ker(T) = \{\langle 0, 0, t\rangle \,|\, t \in \mathbb{C}\}$, and $im(T) = \{\langle t, 2t, t + s\rangle \,|\, t, s \in \mathbb{C}\}$. We just saw that both of these subspaces are T-invariant. However, these two subspaces have an intersection which contains, for example, the vector $\vec{v} = \langle 0, 0, 1\rangle$. Therefore, $\mathbb{C}^3 \neq ker(T) \oplus im(T)$.

Unfortunately, the kernel and image of T are not always complementary invariant subspaces.

Eventual Kernel and Eventual Image

Consider the positive integer powers of T. Observe that if $\vec{v} \in ker(T^k)$ then $\vec{v} \in ker(T^{k+1})$, since if $T^k(\vec{v}) = \vec{0}$ then $T^{k+1}(\vec{v}) = T(T^k(\vec{v})) = T(\vec{0}) = \vec{0}$. Similarly, if $\vec{v} \in im(T^{k+1})$ then $\vec{v} \in im(T^k)$ since if $\vec{v} = T^{k+1}(\vec{w})$ then $\vec{v} = T^k(T(\vec{w}))$. Therefore, we have the following inclusions.

$$ker(T) \subseteq ker(T^2) \subseteq ker(T^3) \subseteq ... \subseteq ker(T^k) \subseteq ker(T^{k+1}) \subseteq ...$$

$$im(T) \supseteq im(T^2) \supseteq im(T^3) \supseteq ... \supseteq im(T^k) \supseteq im(T^{k+1}) \supseteq ...$$

Lemma: If $ker(T^k) = ker(T^{k+1})$, then $ker(T^k) = ker(T^{k+\ell})$ for all $\ell > 0$.

The idea here is quite simple. If $\vec{v} \in ker(T^{k+\ell})$, this means $k + \ell$ applications of T to $\vec{v}$ will yield zero, but $T^{k+\ell}(\vec{v}) = T^{k+1}T^{\ell-1}(\vec{v})$, hence $T^{\ell-1}(\vec{v}) \in ker(T^{k+1})$ and since $ker(T^k) = ker(T^{k+1})$, we conclude that $T^{k+\ell-1}(\vec{v}) = T^kT^{\ell-1}(\vec{v}) = \vec{0}$. This argument can be repeated to yield $T^k(\vec{v}) = \vec{0}$.

$$\overbrace{TTTT}^{k}\,\overbrace{TTTT}^{\ell}(\vec{v}) = \vec{0}$$

$$\overbrace{TTTTT}^{k+1}\,\overbrace{TTT}^{\ell-1}(\vec{v}) = \vec{0}$$

$$\overbrace{TTTT}^{k}\,\overbrace{TTT}^{\ell-1}(\vec{v}) = \vec{0}$$

$$\overbrace{TTTTT}^{k+1}\,\overbrace{TT}^{\ell-2}(\vec{v}) = \vec{0}$$

$$\overbrace{TTTT}^{k}\,\overbrace{TT}^{\ell-2}(\vec{v}) = \vec{0}$$

$$\vdots$$

Exercise: Show that if $im(T^k) = im(T^{k+1})$, then $im(T^k) = im(T^{k+\ell})$ for all $\ell > 0$.

The sequences $\{ker(T^n)\}_{n=1}^{\infty}$ and $\{im(T^n)\}_{n=1}^{\infty}$ are both eventually constant. This is because the kernel and image are both subspaces of U, so if $ker(T^k) \subsetneq ker(T^{k+1})$, we have $dim(ker(T^k)) < dim(ker(T^{k+1}))$. Thus, the sequence $\{ker(T^n)\}_{n=1}^{\infty}$ is strictly increasing until $ker(T^k) = ker(T^{k+1})$, and constant from that point on. For the same reasons, $\{im(T^n)\}_{n=1}^{\infty}$ is eventually constant as well, and hence there is a smallest positive integer m, bounded by the dimension of U, for which $ker(T^m) = ker(T^{m+1})$, $im(T^m) = im(T^{m+1})$, and given any $M > m$, $ker(T^m) = ker(T^M)$ and $im(T^m) = im(T^M)$.

The notion of a properly increasing sequence of nested subspaces comes up often enough that it has a name; mathematicians call it a *flag*. (If you draw a point, then a line containing that point, then a plane which contains that line, it looks like a flag on a flagpole.)

Definition: The *eventual kernel* of T is $ker(T^m)$, and the *eventual image* of T is $im(T^m)$. These subspaces of U will be denoted $evker(T)$ and $evim(T)$. Note that by the remarks above, we could also define the eventual kernel of T as $ker(T^{dim(U)})$ and the eventual image of T as $im(T^{dim(U)})$.

Lemma: Given any linear operator $T : U \to U$, the two subspaces $evker(T)$ and $evim(T)$ are both T-invariant subspaces of U.

This follows very quickly from the definition of the eventual kernel and image. Let $\vec{v} \in evker(T)$. Because $T^m(\vec{v}) = \vec{0}$ and $ker(T^m) = ker(T^{m+1})$, we have $T^m(T(\vec{v})) = T^{m+1}(\vec{v}) = \vec{0}$, and therefore $T(\vec{v}) \in ker(T^m) = evker(T)$. Thus, $evker(T)$ is T-invariant.

Similarly, if $\vec{v} \in evim(T)$ then there is a $\vec{w} \in U$ with $\vec{v} = T^m(\vec{w})$, hence $T(\vec{v}) = T^{m+1}(\vec{w})$, but $im(T^{m+1}) = im(T^m)$, so $T(\vec{v}) \in im(T^m) = evim(T)$ and $evim(T)$ is T-invariant.

Now, we come to the first of our three main results, which provides two complementary invariant subspaces that can be used to decompose any linear operator.

Theorem I: Let $T : U \to U$ be a linear operator. The eventual kernel of T and the eventual image of T are complementary T-invariant subspaces of U. That is, $U = evker(T) \oplus evim(T)$, and T can be decomposed as $T = T|_{evker(T)} \oplus T|_{evim(T)}$.

We have already seen that $evker(T)$ and $evim(T)$ are T-invariant subspaces. Furthermore, by the rank-nullity theorem applied to T^m,

$$dim(evker(T)) + dim(evim(T)) = dim(ker(T^m)) + dim(im(T^m)) = dim(U).$$

To conclude that $U = evker(T) \oplus evim(T)$ and complete the proof, we need to show that $evker(T)$ and $evim(T)$ have no nonzero vectors in their intersection. Let $\vec{v} \in evim(T)$, that is, $\vec{v} = T^m(\vec{w})$. If $\vec{v} \in evker(T)$ also, we have $\vec{0} = T^m(\vec{v}) = T^m(T^m(\vec{w})) = T^{2m}(\vec{w})$. However, $ker(T^{2m})$ and $ker(T^m)$ are equal, by definition of m, so $\vec{w} \in ker(T^m)$. Therefore, $\vec{v} = T^m(\vec{w}) = \vec{0}$. ■

This power of this result becomes clear when it's applied, not to the linear operator T, but to the linear operator $T - \lambda I$, for some eigenvalue λ. In doing this, we can, from any eigenvalue, construct a pair of complementary T-invariant subspaces.

Exercise: Show that a subspace is $T - \lambda I$-invariant iff it is T-invariant.

Example: Let $T : \mathbb{C}^3 \to \mathbb{C}^3$ with $T\langle z_1, z_2, z_3\rangle = \langle -9z_1 - 6z_2 + z_3,\ 12z_1 + 9z_2 - 2z_3,\ z_1 + z_2 + z_3\rangle$.

Let's try to find a decomposition of T, using the theorem above. There are no obvious T-invariant subspaces, but if you look for eigenvalues and eigenvectors, you'll find that $\lambda = 2$ is an eigenvalue, with an eigenvector $\vec{v} = \langle 1, -2, -1\rangle$. Now consider the linear operator $S = T - 2I$. Let's write out the matrix of this operator in the standard basis.

$$[S] = \begin{bmatrix} -11 & -6 & 1 \\ 12 & 7 & -2 \\ 1 & 1 & -1 \end{bmatrix}$$

Since $dim(\mathbb{C}^3) = 3$, we know that $evker(S) = ker(S^3)$ and $evim(S) = im(S^3)$, both of which can be easily calculated by cubing the matrix above.

$$[S^3] = \begin{bmatrix} -250 & -125 & 0 \\ 250 & 125 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, $evker(S) = \{\langle z_1, z_2, z_3\rangle \mid z_2 = -2z_1\}$, and $evim(S) = \{\langle z_1, z_2, z_3\rangle \mid z_2 = -z_1 \text{ and } z_3 = 0\}$.

This leads us to consider the basis $\beta = \{\langle 1, -2, 0\rangle, \langle 0, 0, 1\rangle, \langle 1, -1, 0\rangle\}$ of $\mathbb{C}^3$ obtained by amalgamating a basis for $evker(S)$ and a basis for $evim(S)$. If we write the matrix of T in this basis, then after a little calculation, we obtain the block diagonal matrix below.

$$[T]_\beta = \left[\begin{array}{cc|c} 3 & 1 & 0 \\ -1 & 1 & 0 \\ \hline 0 & 0 & -3 \end{array}\right]$$

We have decomposed T as a direct sum of two linear operators on two invariant (both S-invariant and T-invariant, see the exercise above) subspaces of $\mathbb{C}^3$. The subspace $evker(S) = evker(T - 2I)$ is called the *generalized eigenspace* of T for $\lambda = 2$, and vectors in this space are called *generalized eigenvectors* for $\lambda = 2$.

We now have a general strategy for decomposing linear operators. Find an eigenvalue λ, and then calculate $V = evker(T - \lambda I)$ and $W = evim(T - \lambda I)$. These will be complementary T-invariant subspaces. If we continue by restricting our attention to $evim(T - \lambda I)$ and repeating this process, will we be able to keep going until we have one invariant subspace for each distinct eigenvalue?

Theorem II: Suppose $T : U \to U$ has eigenvalues $\lambda_1, \lambda_2, ..., \lambda_k$. Then U is the direct sum of the generalized eigenspaces of T, that is, $U = evker(T - \lambda_1 I) \oplus evker(T - \lambda_2 I) \oplus ... \oplus evker(T - \lambda_k I)$.

First, let's show the generalized eigenvectors of T span U by induction. Let W be a T-invariant subspace of U, of dimension n. If $n = 1$, then $T|_W$ has exactly one eigenvalue λ, which is also an eigenvalue of T, and W is one dimensional, so $U = ker(T - \lambda I) = evker(T - \lambda I)$.

Suppose that for any invariant subspace W of dimension less than n, the generalized eigenvectors of $T|_W$ span W. Let λ be any eigenvalue of T. We have that $U = evker(T - \lambda I) \oplus evim(T - \lambda I)$ by Th.I, and $evker(T - \lambda I) \neq \{\vec{0}\}$ since it contains a nonzero eigenvector, hence the dimension of $evim(T - \lambda I)$ is less than n. Because $evim(T - \lambda I)$ is T-invariant, the generalized eigenvectors of $T|_{evim(T-\lambda I)}$ span $evim(T - \lambda I)$ by induction, and any generalized eigenvector of $T|_{evim(T-\lambda I)}$ is a generalized eigenvector of T. The subspace $evker(T - \lambda I)$ is spanned by generalized eigenvectors of T by definition, so $U = evker(T-\lambda I) \oplus evim(T-\lambda I)$ is spanned by generalized eigenvectors of T.

Next, we need to show that the generalized eigenspaces for distinct eigenvalues intersect only at $\vec{0}$. Suppose that $\vec{v} \in evker(T - \lambda_1 I) \cap evker(T - \lambda_2 I)$ is nonzero. After some number of applications (say p) of $T - \lambda_1 I$ to $\vec{v}$, the result is zero, but this means $(T - \lambda_1 I)^{p-1}(\vec{v})$ (which is nonzero, by definition of p) is an eigenvector of T, not just a generalized eigenvector.

$$\begin{aligned} (T - \lambda_1 I)(T - \lambda_1 I) \; ... \; (T - \lambda_1 I)(\vec{v}) &= \vec{0} \\ (T - \lambda_1 I)((T - \lambda_1 I)^{p-1}(\vec{v})) &= \vec{0} \end{aligned}$$

Moreover, this vector $(T - \lambda_1 I)^{p-1}(\vec{v})$ is still in $evker(T - \lambda_2 I)$. Why? We began by assuming that $\vec{v}$ is in $evker(T - \lambda_2 I)$, hence so is its scalar multiple $\lambda_1 \vec{v}$. $T(\vec{v})$ is also in there, because generalized eigenspaces are T-invariant. Thus, the difference $T(\vec{v}) - \lambda_1 \vec{v} = (T - \lambda_1 I)(\vec{v})$ is in $evker(T - \lambda_2 I)$ as well. This argument shows that applying $T - \lambda_1 I$ to any vector in $evker(T - \lambda_2 I)$ always yields another vector in $evker(T - \lambda_2 I)$, and hence $(T - \lambda_1 I)^{p-1}(\vec{v}) \in evker(T - \lambda_2 I)$.

Let $\vec{w} = (T - \lambda_1 I)^{p-1}(\vec{v})$. We've seen $\vec{w}$ is a nonzero eigenvector of T with eigenvalue λ_1 which is in $evker(T - \lambda_2 I)$. If we apply $T - \lambda_2 I$ to $\vec{w}$, then $T(\vec{w}) - \lambda_2 I(\vec{w}) = \lambda_1 \vec{w} - \lambda_2 \vec{w} = (\lambda_1 - \lambda_2)\vec{w}$. Thus, applying $T - \lambda_2 I$ to $\vec{w}$ multiplies it by the nonzero scalar $\lambda_1 - \lambda_2$, but then $\vec{w} \notin evker(T - \lambda_2 I)$, a contradiction. Hence, there cannot be a nonzero $\vec{v} \in evker(T - \lambda_1 I) \cap evker(T - \lambda_2 I)$. ■

Therefore, a linear operator with k distinct eigenvalues will have a decomposition with k invariant subspaces (the generalized eigenspaces), and hence can be written as a block diagonal matrix with k blocks. On each of the generalized eigenspaces, the restriction of T will have only one eigenvalue, so what can we say about a linear operator with only one eigenvalue?

Nilpotent Operators & Orbits

Example: Consider the operator $T : \mathbb{C}^3 \to \mathbb{C}^3$ given by $T\langle z_1, z_2, z_3\rangle = \langle z_1 + z_2 + z_3,\ z_2 + z_3,\ z_3\rangle$.

We studied this example earlier and saw $T(\langle 1, 0, 0\rangle) = \langle 1, 0, 0\rangle$, so $\lambda = 1$ is an eigenvalue of T. In fact, it's the only eigenvalue of T. So, consider the operator $S = T - I$, whose matrix in the standard basis is given below.

$$[S] = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

It's easy to see that $[S^3]$ is a zero matrix, and therefore $evker(S) = \mathbb{C}^3$ and $evim(S) = \{\vec{0}\}$. Note that $evker(S) = \mathbb{C}^3$ means precisely that S is nilpotent.

In fact, if T is any linear operator with only one eigenvalue λ, then $S = T - \lambda I$ must be nilpotent. Why? If T has λ as its only eigenvalue, then by Th.II above, $U = evker(S)$, so S is nilpotent.

Lemma: Let $S : U \to U$ be a nilpotent linear operator, and let $\vec{v}$ be a vector with $S^k(\vec{v}) = \{\vec{0}\}$ but $S^{k-1}(\vec{v}) \neq \{\vec{0}\}$. Then the set $\beta = \{\vec{v}, S(\vec{v}), ..., S^{k-1}(\vec{v})\}$ is linearly independent.

If $c_0\vec{v} + c_1 S(\vec{v}) + ... + c_{k-1}S^{k-1}(\vec{v}) = \vec{0}$, then by repeated applications of S, we can force all but one term to be zero.

$$\begin{aligned} c_0\vec{v} + c_1 S(\vec{v}) + ... + c_{k-1}S^{k-1}(\vec{v}) &= \vec{0} \\ c_0 S^{k-1}(\vec{v}) + c_1 S^k(\vec{v}) + ... + c_{k-1}S^{2k-1}(\vec{v}) &= \vec{0} \\ c_0 S^{k-1}(\vec{v}) + c_1\vec{0} + ... + c_{k-1}\vec{0} &= \vec{0} \\ c_0 S^{k-1}(\vec{v}) &= \vec{0} \end{aligned}$$

Thus, $c_0 = 0$, because $S^{k-1}(\vec{v}) \neq \{\vec{0}\}$ by assumption. Now that $c_0 = 0$ we can apply S^{k-2} to show $c_1 = 0$. Repeating this argument k times will show each c_i must be zero.

$$\begin{aligned} c_0\vec{v} + c_1 S(\vec{v}) + ... + c_{k-1}S^{k-1}(\vec{v}) &= \vec{0} \\ \vec{0} + c_1 S^{k-1}(\vec{v}) + c_2 S^k(\vec{v}) + ... + c_{k-1}S^{2k-2}(\vec{v}) &= \vec{0} \\ c_1 S^{k-1}(\vec{v}) + c_2\vec{0} + ... + c_{k-1}\vec{0} &= \vec{0} \\ c_1 S^{k-1}(\vec{v}) &= \vec{0} \end{aligned}$$

Definition: The set $\beta = \{\vec{v}, S(\vec{v}), S^2(\vec{v}), ..., S^{m-1}(\vec{v})\}$ is the *orbit of* $\vec{v}$ *under* S, and the subspace spanned by this orbit is called the S*-cyclic subspace generated by* $\vec{v}$.

Exercise: Show that the S-cyclic subspace generated by any vector $\vec{v}$ is S-invariant.

Example: Consider the operator $T : \mathbb{C}^3 \to \mathbb{C}^3$ given by $T\langle z_1, z_2, z_3\rangle = \langle z_1 + z_2 + z_3,\ z_2 + z_3,\ z_3\rangle$.

What does the lemma above tell us about T? In the last example, we defined $S = T - I$, and found $S^3 = 0$. To apply the lemma, we need a vector $\vec{v}$ such that $S^2(\vec{v}) \neq \vec{0}$.

$$[S^2] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, $\vec{v} = \langle 0, 0, 1\rangle$ is such a vector. Note that $S(\vec{v}) = \langle 1, 1, 0\rangle$, $S^2(\vec{v}) = \langle 1, 0, 0\rangle$, and $S^3(\vec{v}) = \langle 0, 0, 0\rangle$, so the set $\beta = \{S^2(\vec{v}), S(\vec{v}), \vec{v}\}$ is indeed linearly independent, as we saw in the lemma above. In this case, the S-cyclic subspace generated by $\vec{v}$ is all of $\mathbb{C}^3$. Observe that S acts on the basis β in a very simple way. It sends the leftmost basis vector to zero, and shifts the others one position left.

$$[S]_\beta = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Note $\beta = \{S^2(\vec{v}), S(\vec{v}), \vec{v}\}$ is ordered with higher powers of S on the left, so that the matrix above is upper triangular. In the opposite order, $[S]_\beta$ will be lower triangular. What is the matrix of T written in this basis? Well, $S = T - I$, so $T = S + I$.

$$[T]_\beta = [S]_\beta + [I]_\beta = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

At this point we can see the geometric significance of Jordan cells. Every Jordan cell is a sum of two actions on a T-invariant subspace: A 'left shift' which corresponds a vector being pushed along its orbit under $S = T - \lambda I$, and a multiple of the identity which corresponds to scaling by a factor of λ.

Theorem III: Let $S : U \to U$ be a nilpotent linear operator. There is a basis for U which is made up of orbits of vectors under S.

We can easily find a set of orbits that span U. Take any basis $\beta = \{v_1, v_2, ..., v_n\}$ of U, and for each v_i, repeatedly apply S until the result is zero, and add these vectors to β (the orbits have finite length since S is nilpotent). The result will span U, but in general will not be linearly independent.

Now we need to remove vectors from our inflated basis β to whittle it back down to being linearly independent, while maintaining the key property of β, that it is a set of orbits, or equivalently, it is closed under S. The notation gets a bit cumbersome if we try to do this in general, so let's suppose that $\beta = \{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ is a basis for U, and after adding in the orbits of each of these vectors, we obtain $\beta = \{\vec{v_1}, S(\vec{v_1}), S^2(\vec{v_1}), \vec{v_2}, S(\vec{v_2}), \vec{v_3}, S(\vec{v_3}), S^2(\vec{v_3}), S^3(\vec{v_3})\}$. It will be clear that the process used to reduce β to a linearly independent set works in general.

If the set β is not linearly independent, there must be some linear relation between the vectors in β. Let's take $aS(\vec{v_1}) + bS(\vec{v_2}) + cS^2(\vec{v_3}) = \vec{0}$ as our linear relation. Now, apply S as many times as we can before everything vanishes. This will reduce any linear relation to one involving only the final vectors in the orbits. In this case, we obtain $aS^2(\vec{v_1}) + bS^3(\vec{v_3}) = \vec{0}$.

Factoring out the largest possible power of S, we obtain $S^2(a\vec{v_1}+bS(\vec{v_3})) = \vec{0}$. The result is always a power of S (possibly zero) applied to a linear combination in which at least one of the initial vectors in an orbit occurs, since if not, we could factor out a larger power of S. We now replace the orbit of that initial vector, in this case $\{\vec{v_1}, S(\vec{v_1}), S^2(\vec{v_1})\}$, with the orbit $\{a\vec{v_1} + bS(\vec{v_3}), S(a\vec{v_1} + bS(\vec{v_3}))\}$. This shortens one of the orbits, but leaves the span of β unchanged. Why? Note that the final vector which was removed can be written as a linear combination of the other vectors in β, by the linear relation at the end of the preceding paragraph. Furthermore, note that the second vector in the new orbit, $S(a\vec{v_1}+bS(\vec{v_3}))$, can be written as $aS(\vec{v_1})+bS^2(\vec{v_3})$, and since that vector and $S^2(\vec{v_3})$ are in β, $S(\vec{v_1})$ is still in the span of β. In this way, all vectors in the orbit which was removed will still be in the span of β.

Note that it's possible we will end up removing an orbit completely. For example, if $v_i \in \beta$ had an orbit of length one, and we had a linear relation $\vec{v_i} + aS(\vec{v_j}) + bS^2(\vec{v_k}) = \vec{0}$. In this case, if we follow the procedure above, we would simply remove v_i. In general, any linear relation can be used to reduce the number of vectors in β by one by replacing one of the orbits with another of shorter length, while preserving $span(\beta)$. This 'whittling down' process can be repeated until no linear relations exist between vectors in β, at which point β is a basis for U. ■

Exercise: Write a rigorous proof of Th.III. (Exercise in good notation & patience.)

With Th.III verified, we have now have a complete argument for the existence of the Jordan normal form of any linear operator T on $\mathbb{C}^n$. Let's review the work we did to get here.

- T can be written as a direct sum of linear operators on generalized eigenspaces, which are invariant subspaces of the form $evker(T-\lambda I)$ (by Th.II). This means the matrix of T is block diagonal, with one block per generalized eigenspace.
- On the generalized eigenspace for the eigenvalue λ, $S = T - \lambda I$ is nilpotent (by Th.II), hence there is a basis β for $evker(T - \lambda I)$ consisting of orbits of vectors under S (by Th.III).
- The matrix of S in the basis β (ordered with vectors in the same orbit grouped together, and higher powers of S appearing first in each orbit) has $[S]_{i,i+1} = 1$ when the i-th and $(i+1)$-th vector in β are in the same orbit, and all other entries are zeros. The matrix of $T|_{ker(T-\lambda I)}$ in this basis is then obtained by placing λ's on the diagonal.

The indecomposable operators that T is built up from, which got us started on this quest in the first place, can now be clearly seen. They are the restrictions of T to each S-cyclic subspace, that is, the restriction of T to the span of an orbit. These are known as *Jordan cells*.

Exercise: Show that the restriction of T to a Jordan cell is indecomposable. (*Hint:* First reduce the problem to showing the restriction of $S = T - \lambda I$ to a Jordan cell is indecomposable, and then consider a vector in $ker(S^m)$ outside of $ker(S^{m-1})$, where m is the length of the orbit defining the Jordan cell.)

Example: Consider $T : \mathbb{C}^4 \to \mathbb{C}^4$ defined by the matrix below (written in the standard basis for $\mathbb{C}^4$). This linear operator has two distinct eigenvalues, $\lambda_1 = 2$ and $\lambda_2 = 3$.

$$[T] = \begin{bmatrix} 2 & -1 & 0 & 1 \\ 0 & 3 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 3 \end{bmatrix}$$

Let's use our results above to write T in Jordan normal form. Since $\lambda_1 = 2$, we should consider $S = T - 2I$. Since S operates on a space of dimension four, we have $evker(S) = ker(S^4)$.

$$[S] = \begin{bmatrix} 0 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \qquad [S^4] = \begin{bmatrix} 0 & -2 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & 1 \end{bmatrix}$$

So $evker(S) = \{\langle z_1, z_2, z_3, z_4\rangle \,|\, 2z_2 = z_3 + z_4\}$, $evim(S) = \{\langle z_1, z_2, z_3, z_4\rangle \,|\, z_1 = z_4, z_2 = 0, z_3 = 0\}$. Take $\{\langle 1,0,0,0\rangle, \langle 0,1,2,0\rangle, \langle 0,1,0,2\rangle\}$ as a basis for $evker(S)$, $\{\langle 1,0,0,1\rangle\}$ as a basis for $evim(S)$, and amalgamate them to form $\beta = \{\langle 1,0,0,0\rangle, \langle 0,1,2,0\rangle, \langle 0,1,0,2\rangle, \langle 1,0,0,1\rangle\}$. The matrix of T is this basis has two blocks, and is given below.

$$[T]_\beta = \left[\begin{array}{ccc|c} 2 & -1 & 1 & 0 \\ 0 & \frac{3}{2} & \frac{5}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ \hline 0 & 0 & 0 & 3 \end{array}\right]$$

But we can do better, and get the Jordan normal form. By Theorem III, we should be able to find a basis of $evker(S)$ composed of orbits under T. To do this, we look at smaller powers of S.

$$[S^2] = \begin{bmatrix} 0 & -2 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & 1 \end{bmatrix} \qquad [S^3] = \begin{bmatrix} 0 & -2 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & 1 \end{bmatrix}$$

Thus, $evker(S) = ker(S^2)$, which has dimension three, and $ker(S) = \{\langle z_1, z_2, z_3, z_4\rangle \,|\, z_2 = z_3 = z_4\}$, which has dimension two. Therefore, there must be a basis of $evker(S)$ consisting of two orbits, $\{\vec{v_1}, S(\vec{v_1})\}$ and $\{\vec{v_2}\}$, where $\vec{v_1} \in ker(S^2)$ but not $ker(S)$, and $\vec{v_2} \in ker(S)$ is not in $span(\vec{v_1}, S(\vec{v_1}))$.

Examining $ker(S)$ and $ker(S^2)$, we can find suitable vectors $\vec{v_1}$ and $\vec{v_2}$. Take $\vec{v_1} = \langle 0,1,2,0\rangle$, then $S(\vec{v_1}) = \langle -1,-1,-1,-1\rangle$. For $\vec{v_2}$, we need a vector in $ker(S)$ which is not in $span(\vec{v_1}, S(\vec{v_1}))$. The vector $\vec{v_2} = \langle 1,0,0,0\rangle$ is an easy choice. Thus, $\{\langle -1,-1,-1,-1\rangle, \langle 0,1,2,0\rangle, \langle 1,0,0,0\rangle\}$ is a basis for $evker(S)$ (with orbits grouped together, and higher powers of S on the left), and putting this together with $evim(S)$, we obtain $\gamma = \{\langle -1,-1,-1,-1\rangle, \langle 0,1,2,0\rangle, \langle 1,0,0,0\rangle, \langle 1,0,0,1\rangle\}$, a basis in which T is written as a direct sum of Jordan cells.

$$[T]_\gamma = \left[\begin{array}{cc|c|c} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ \hline 0 & 0 & 2 & 0 \\ \hline 0 & 0 & 0 & 3 \end{array}\right]$$

Exercise: Suppose $T : \mathbb{C}^4 \to \mathbb{C}^4$ is a linear operator such that $S = T + 3I$ is nilpotent, $ker(S)$ has dimension one, $ker(S^2)$ has dimension three, and $ker(S^3)$ has dimension four. How many orbits are in the Jordan basis for T? Write the Jordan normal form of T.

Exercise: Consider $T : \mathbb{C}^3 \to \mathbb{C}^3$ defined by $T\langle z_1, z_2, z_3\rangle = \langle z_1 + z_2 + z_3,\ z_2 + z_3,\ z_3\rangle$. We have studied this example extensively, and noted that $ker(S^2) \neq \mathbb{C}^3$ for $S = T - I$. Use this fact to prove that T must be indecomposable.

Constructing a Jordan Basis

In the last example, it was easy enough find a basis of orbits for S by following our nose, and we know such a basis always exists by Th. III. But when it comes to constructing one, no general strategy was given. It would be nice to have some procedure which, given any nilpotent operator $S : U \to U$, will build a basis of orbits.

To illustrate why this is not such a simple task, suppose we have a nilpotent operator $S : U \to U$ for which $ker(S^4) = U$ and $dim(ker(S^4)) = dim(ker(S^3)) + 2$. If we take a vector $\vec{v} \in ker(S^4)$ which is not in $ker(S^3)$, then we know that the orbit $\{\vec{v}, S(\vec{v}), \ldots, S^3(\vec{v})\}$ is a linearly independent set. The problem arises when we try to add another orbit. We need one more vector $\vec{w} \in ker(S^4)$ which is not in $ker(S^3)$, but how should we pick $\vec{w}$ so that the union of the two orbits $\{\vec{v}, S(\vec{v}), \ldots, S^3(\vec{v})\}$ and $\{\vec{w}, S(\vec{w}), \ldots, S^3(\vec{w})\}$ is linearly independent?

Even if we chose $\vec{w}$ so that $\{\vec{v}, \vec{w}\}$ is linearly independent, it may be $\vec{w} = \vec{v} + \vec{u}$, where, for example, $S^2(\vec{u}) = 0$. If this happens, $S^2(\vec{w}) = S^2(\vec{v} + \vec{u}) = S^2(\vec{v}) + S^2(\vec{u}) = S^2(\vec{v})$, and the two orbits collide. We need some way of ensuring our $\vec{w} \in ker(S^4)$ does not differ from $\vec{v}$ by some $\vec{u} \in ker(S^k)$ for any $k < 4$. We can solve this problem by identifying vectors in $ker(S^4)$ that differ by an element of $ker(S^3)$, i.e., by working in the quotient space $ker(S^4)/ker(S^3)$.

Let $S : U \to U$ be a nilpotent linear operator on a finite dimensional vector space. To construct a basis β of U composed of orbits under S, i.e., a Jordan basis, we can do the following.

1. Find the least m such that $ker(S^m) = U$, and consider the space $Q_m = ker(S^m)/ker(S^{m-1})$ as well as the quotient map $\phi_m : ker(S^m) \to Q_m$.

2. Find a basis of Q_m, and take preimages under ϕ_m to obtain $\alpha_m = \{v_1, v_2, ..., v_k\} \in ker(S^m)$. Let $\beta_m = \alpha_m$.

3. Consider $Q_{m-1} = ker(S^{m-1})/ker(S^{m-2})$ and the quotient map $\phi_{m-1} : ker(S^{m-1}) \to Q_{m-1}$. Let $\beta_{m-1} = \{S(v_1), S(v_2), \ldots, S(v_k)\}$ be obtained by applying S to each vector in β_m. Find the image of each vector in β_{m-1} under ϕ_{m-1}, and extend the set so obtained to a basis of Q_{m-1}. For every new vector added in this extension, find a preimage for it under ϕ_{m-1}, and call these preimages $\alpha_{n-1} = \{w_1, w_2, \ldots, w_k\}$. Enlarge β_{m-1} by adding these preimages, to obtain $\beta_{m-1} = \{S(v_1), S(v_2), \ldots, S(v_k), w_1, w_2, \ldots, w_k\}$.

4. Repeat step 3 above for Q_{m-2}, Q_{m-3}, and so on, until β_1 has been constructed and extended. At this point, $\beta = \alpha_1 \cup \alpha_2 \cup \ldots \cup \alpha_m$ will be the desired (but unordered) basis of orbits.

As we saw in the example above, this procedure is typically not necessary when the dimension of U is small, but now we can rest assured that it's possible to constructively produce a Jordan basis for any linear operator T on any complex vector space U.